# INSTRUCTIONS

for the use of

# A SLIDE RULE FOR RADIATION CALCULATIONS

M. W. MAKOWSKI

**Manufactured by A. G. THORNTON LTD.,**
**Wythenshawe, Manchester, England.**

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# No. F5100 RADIATION SLIDE RULE

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## A Slide Rule for Radiation Calculations

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**A slide rule has been developed to facilitate rapid calculations based on the Planck radiation formula. Quantities such as the radiant flux density in a given wave-length region, the spectral radiant flux density at a given wave-length, or the corresponding quantities expressed in photon units, can be obtained readily for a black body over a range $\lambda T = 2\times 10^2$ to $\lambda T = 4\times 10^6$ micron degrees with an accuracy of about 1 percent. Simple extension rules can be used for larger values of $\lambda T$.**

### I. INTRODUCTION

IN the course of certain radiation calculations it was found that the available tables based on the Planck radiation formula were neither sufficiently comprehensive nor convenient for frequent reference. The need was felt for a device that would facilitate rapid calculations without involving too great a sacrifice in accuracy. Accordingly it was decided to develop a nomogram, based on the Wien displacement law, capable of giving rapidly approximate values for the radiation quantities most frequently encountered. Typical of such quantities is the radiation per unit area emitted by a black body at a given temperature in a given wave-length interval. The general features of the nomogram are shown in Figs. 1 and 2.

As the work proceeded the advantages of a slide rule arrangement became apparent, and it was decided to extend the computations already carried out with a view to the development of a radiation slide rule. This paper gives a description of the slide rule and the methods used in the associated calculations. The final form of the rule, the general arrangement of which can be seen in Fig. 3, includes the scales required for calculations involving spectral radiant flux density and radiant flux density, together with the corresponding quantities in photon units. The over-all dimensions are $19\frac{1}{2}$ in.$\times 3\frac{1}{2}$ in. The stock carries the scales of maximum spectral radiant flux density, $H_\lambda$max.,[a] total radiant flux density, $H$, maximum spectral photon flux density, $Q_\lambda$max., total photon flux density, $Q$, temperature (centigrade and absolute), and wave-length. One side of the slide carries the scales required for calculations involving the first two of the above quantities, while the scales necessary for photon calculations are on the

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[a] The symbols used throughout this paper have the usual significance. The subscripts $\lambda$, $\Delta\lambda$, $\nu$, and $\Delta\nu$, refer to wave-length, wave-length interval, wave number, and wave number interval respectively. The symbol *cm* is used to denote unit of length referring to the surface of a black body. The values of the constants used follow Birge ($c_1 = 2\pi hc^2 = 3.7403\times 10^{-12}$ watt cm²; $c_1' = 2\pi c = 1.88355\times 10^{11}$ cm sec.$^{-1}$; $c_2 = hc/k = 1.43848$ cm deg. A).

reverse of the slide. The back of the stock carries brief definitions and instructions, together with the scales relating wave-length, wave number, and electron volts.

## II. THE PLANCK RADIATION FORMULA

If we consider unit area of a black body at temperature $T$ and choose some wave-length $\lambda$, then the radiated energy for a wave-length interval 1 $\mathrm{cm}_{\Delta\lambda}$,[b] which we shall call the spectral radiant flux density, is given by the Planck radiation formula as

$$H_\lambda = \frac{c_1\lambda^{-5}}{e^{c_2/\lambda T}-1} \text{ watt } cm^{-2}\ \mathrm{cm}_{\Delta\lambda}{}^{-1}.$$

The corresponding expression for wave number interval $\Delta\nu = 1\ \mathrm{cm}_{\Delta\nu}{}^{-1}$ is

$$H_\nu = \frac{c_1\nu^3}{e^{c_2\nu/T}-1} \text{ watt } cm^{-2}\ (\mathrm{cm}_{\Delta\nu}{}^{-1})^{-1}.$$

For a wave-length interval $d\lambda$ the energy radiated is $H_\lambda d\lambda$, and for a wave-length interval between $\lambda_1$ and $\lambda_2 (\lambda_1 < \lambda_2)$

$$H_{\lambda_1-\lambda_2} = \int_{\lambda_1}^{\lambda_2} H_\lambda d\lambda = \int_0^{\lambda_2} H_\lambda d\lambda - \int_0^{\lambda_1} H_\lambda d\lambda$$

$$= \int_{\lambda_1}^{\infty} H_\lambda d\lambda - \int_{\lambda_2}^{\infty} H_\lambda d\lambda \text{ watt } cm^{-2}.$$

For a given temperature $T$, the wave-length $\lambda_m$ for which $H_\lambda$ is a maximum, is obtained from $dH_\lambda/d\lambda = 0$, which gives, of course, one case of Wien's displacement law,

$$\lambda_m = 0.28972/T\ \mathrm{cm}_\lambda.$$

By substitution

$$H_\lambda \text{max.} = 1.2875 \times 10^{-11} T^5 \text{ watt } cm^{-2}\ \mathrm{cm}_{\Delta\lambda}{}^{-1}.$$

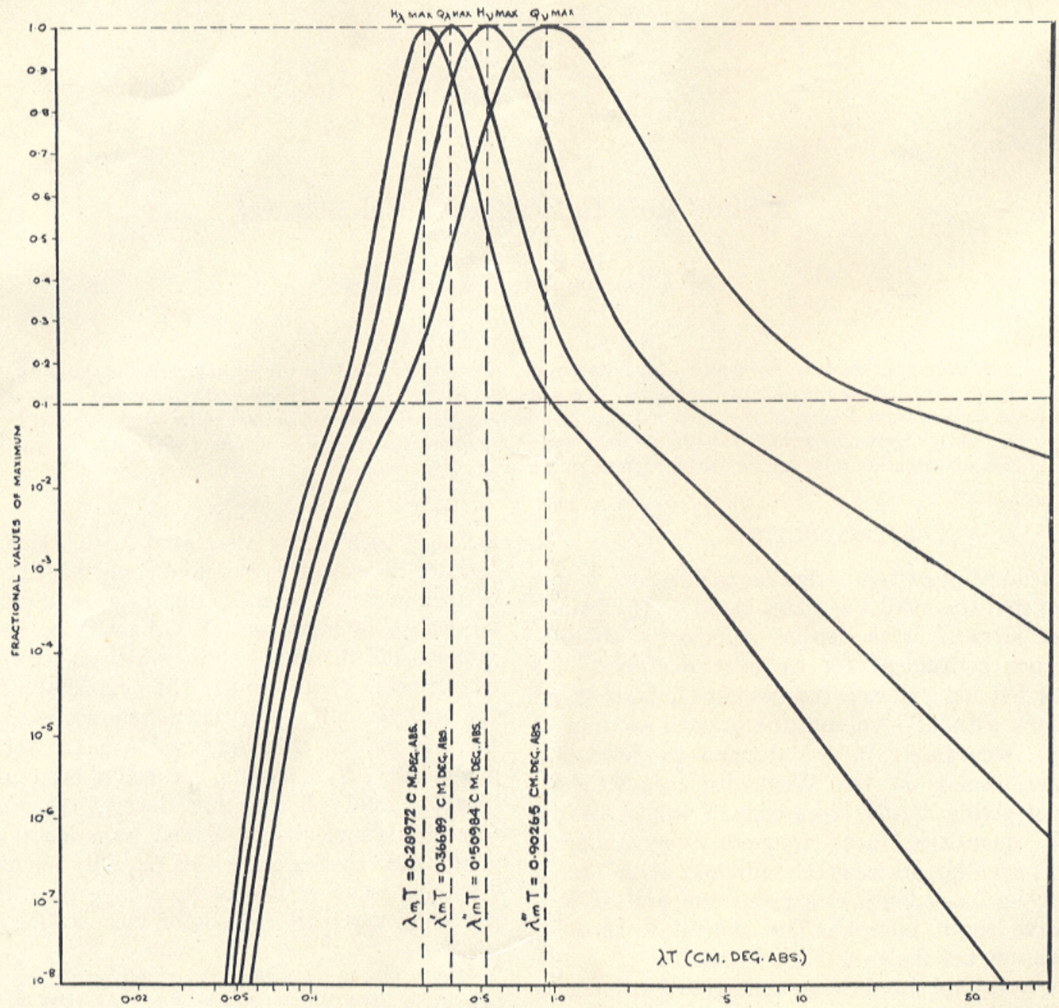


FIG. 1. Radiation nomogram—spectral curves.

[b] It is not possible to attach physical significance to the use of these wave-length and wave number intervals. The spectral quantities refer to a single point on a radiation curve and in all calculations involving a real wave-length interval, the centimeter interval disappears.

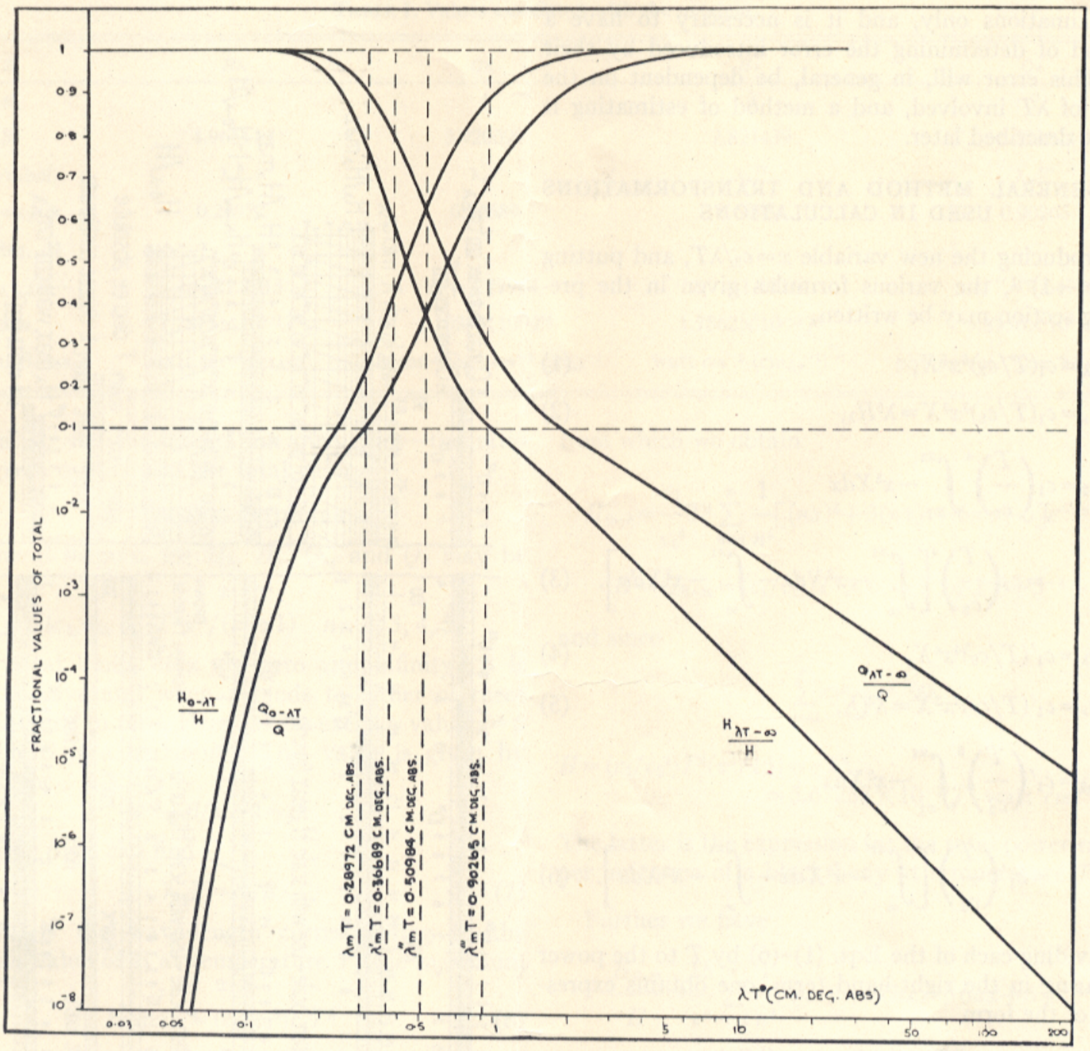


FIG. 2. Radiation nomogram—integration curves.

The total radiant flux density, $H$, is obtained by integrating $H_\lambda$ over the wave-length range from zero to infinity, which process leads to the Stefan-Boltzmann law of radiation,

$$H = H_{0-\infty} = 5.67283 \times 10^{-12} T^4 = \sigma T^4 \text{ watt } cm^{-2}.$$

Since the energy associated with one photon is $hc/\lambda$, the corresponding formula for spectral photon flux density is

$$Q_\lambda = \frac{c_1' \lambda^{-4}}{e^{c_2/\lambda T} - 1} \text{ photon sec.}^{-1}\, cm^{-2}\, \text{cm}_{\Delta\lambda}^{-1}.$$

The remaining formulas may be obtained in terms of photons:

$$Q_\nu = \frac{c_1' \nu^2}{e^{c_2 \nu / T} - 1} \text{ photon sec.}^{-1}\, cm^{-2}\, (\text{cm}_{\Delta\nu}^{-1})^{-1},$$

$$Q_{\lambda_1 - \lambda_2} = \int_{\lambda_1}^{\lambda_2} Q_\lambda d\lambda = \int_0^{\lambda_2} Q_\lambda d\lambda - \int_0^{\lambda_1} Q_\lambda d\lambda$$

$$= \int_{\lambda_1}^{\infty} Q_\lambda d\lambda - \int_{\lambda_2}^{\infty} Q_\lambda d\lambda, \text{ photon sec.}^{-1}\, cm^{-2},$$

$$\lambda_m' = 0.366895/T \text{ cm}_\lambda,$$

$$Q_\lambda \text{ max.} = 2.1027 \times 10^{11} T^4 \text{ photon sec.}^{-1}\, cm^{-2}\, \text{cm}_{\Delta\lambda}^{-1},$$

$$Q = Q_{0-\infty} = 1.5213 \times 10^{11} T^3 = \sigma' T^3 \text{ photon sec.}^{-1}\, cm^{-2}.$$

If the radiating body obeys Lambert's law, quantities associated with the radiation in unit solid angle normal to the radiating surface may be obtained by dividing the above formulas by $\pi$. To avoid multiplicity of scales such quantities have not been included on the rule.

For shorter or longer wave-lengths the simple Wien or Rayleigh-Jeans formulas may be applied. These are

approximations only, and it is necessary to have a method of determining the error introduced by their use. This error will, in general, be dependent on the range of $\lambda T$ involved, and a method of estimating it will be described later.

### III. GENERAL METHOD AND TRANSFORMATIONS USED IN CALCULATIONS

Introducing the new variable $x = c_2/\lambda T$, and putting $X = (e^x - 1)^{-1}$, the various formulas given in the preceding section may be written,

$$H_\lambda = c_1(T/c_2)^5 x^5 X, \tag{1}$$

$$H_\nu = c_1(T/c_2)^3 x^3 X = \lambda^2 H_\lambda, \tag{2}$$

$$H_{\lambda_1-\lambda_2} = c_1\left(\frac{T}{c_2}\right)^4 \int_{x_1}^{x_2} -x^3 X dx$$

$$= c_1\left(\frac{T}{c_2}\right)^4 \left[\int_\infty^{x_2} -x^3 X dx - \int_\infty^{x_1} -x^3 X dx\right], \tag{3}$$

$$Q_\lambda = c_1'(T/c_2)^4 x^4 X, \tag{4}$$

$$Q_\nu = c_1'(T/c_2)^2 x^2 X = \lambda^2 Q_\lambda, \tag{5}$$

$$Q_{\lambda_1-\lambda_2} = c_1'\left(\frac{T}{c_2}\right)^3 \int_{x_1}^{x_2} -x^2 X dx$$

$$= c_1'\left(\frac{T}{c_2}\right)^3 \left[\int_\infty^{x_2} -x^2 X dx - \int_\infty^{x_1} -x^2 X dx\right]. \tag{6}$$

On dividing each of the Eqs. (1)–(6) by $T$ to the power appearing in the right-hand term, one obtains expressions of the form

$$BT^{-n} = f_1(c_1, c_2, c_1')f(x)$$

of which $n$ takes the values 2, 3, 4, 5, and $B$ is in order $H_\lambda$, $H_\nu$, etc. $f_1$ is a function of the physical quantities only, and $f(x)$, treated as a function of the dimensionless variable $x$, is a purely mathematical expression which is specific for each quantity and independent of the chosen values of the physical constants.

From the general expression it can be seen that (a) Since each expression of the form $BT^{-n}$ is dependent only on some function of $x = c_2/\lambda T$, it is possible to find for one particular value of $BT^{-n}$ a set of pairs of $\lambda$ and $T$ such that $\lambda_1 T_1 = \lambda_2 T_2 = \lambda_3 T_3 = \cdots = \lambda_n T_n = c_2/x$. This is the general form of the Wien displacement law. If we take any maximum value of spectral quantity $B_m T^{-n}$ we get $\lambda_m T = c_2/\beta = \text{const.}$, the form in which Wien's law is usually expressed. (b) In order to obtain the value of $B$ for given values of $\lambda$ and $T$, it is necessary to calculate $x = c_2/\lambda T$ and $BT^{-n}$ from the product of $f_1(c_1, c_2, c_1')$ and $f(x)$ when finally $(BT^{-n})T^n = B$.

In the slide rule fractional values of the quantities are introduced. These are, for spectral quantities—the ratio of the given quantity to the corresponding maxi-

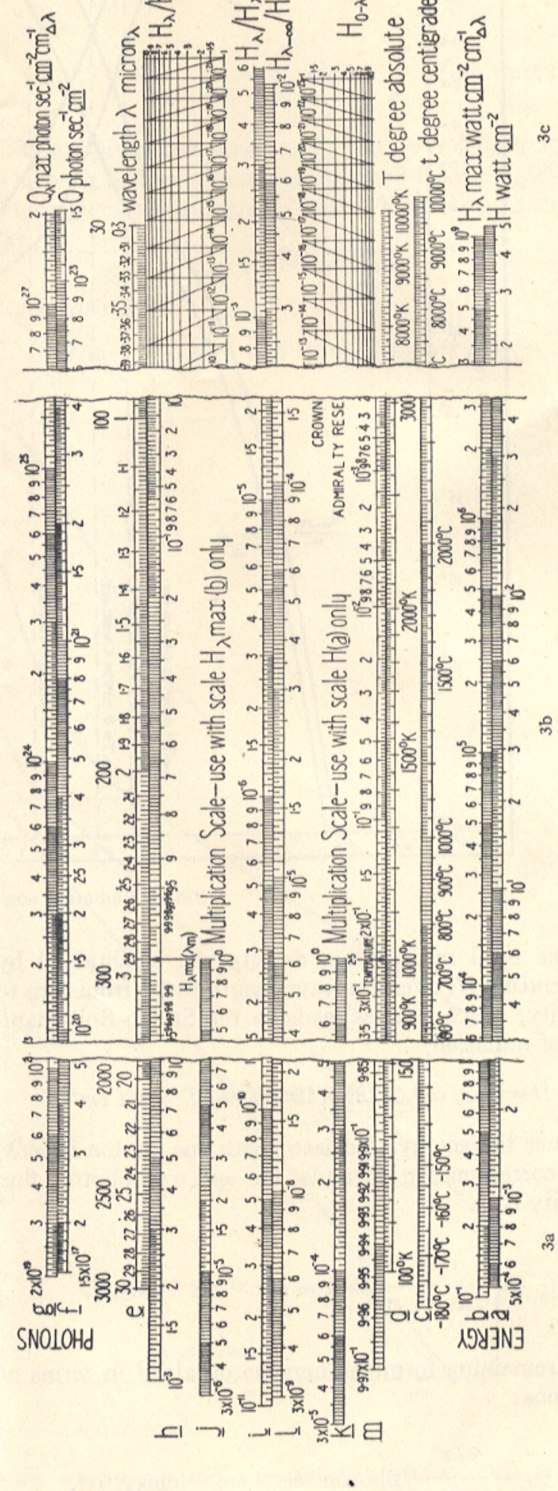


FIG. 3. Front of stock and energy side of slide. Only the ends and a central portion of the slide rule are shown. Scale $i$ and $l$ together with the corresponding longer wave-length figures on scale $e$ are colored red.

TABLE I. Values associated with the maxima.

| $B$ | $H\lambda$ | $Q\lambda$ | $H\nu$ | $Q\nu$ |
|---|---|---|---|---|
| $n$ | 5 | 4 | 3 | 2 |
| $\beta$ | 4.965114 | 3.920690 | 2.821439 | 1.593624 |
| $\lambda_m$(symbol) | $\lambda_m$ | $\lambda_m'$ | $\lambda_m''$ | $\lambda_m'''$ |
| $\lambda_m T=c_2/\beta$ (cm deg. A) | 0.28972 | 0.36689 | 0.50984 | 0.90265 |
| Max. of $x^n/(e^x-1)$ | $c_4=21.2013$ | $c_4'=4.77984$ | $c_4''=1.42144$ | $c_4'''=0.64761$ |
| $B$max. | $1.2875\times10^{-11}T^5$ | $2.1027\times10^{11}T^4$ | $1.7862\times10^{-12}T^3$ | $5.8950\times10^{10}T^2$ |
| Units of $B$max. | watt $cm^{-2}$ $\mathrm{cm}_{\Delta\lambda}^{-1}$ | photon sec.$^{-1}$ $cm^{-2}$ $\mathrm{cm}_{\Delta\lambda}^{-1}$ | watt $cm^{-2}$ $(\mathrm{cm}_{\Delta\nu}^{-1})^{-1}$ | photon sec.$^{-1}$ $cm^{-2}$ $(\mathrm{cm}_{\Delta\nu}^{-1})^{-1}$ |

mum value, and for integration quantities—the ratio of the given quantity to the total value.

### 1. Spectral formulas

A general formula for $H_\lambda$, $H_\nu$, $Q_\lambda$, and $Q_\nu$ may be written

$$B=f_1(c_1, c_2, c_1')T^n x^n/(e^x-1) \quad n=2, 3, 4, 5.$$

The physical limits of $x$ are zero and infinity; $B$ is always positive and when $x$ tends to either of these limits, $B$ tends to zero. Hence there exists a value $x=\beta$ for which $B$ is a maximum. This value is given by $dB/dx=0$ or

$$e^{-\beta}+\beta/n-1=0.$$

Knowing $\beta$ we can find

$$\lambda_m T=c_2/\beta \tag{7}$$

where $\lambda_m$ is the wave-length corresponding to the maximum value of $B$ at temperature $T$; the maximum value of $B$

$$H_\lambda\text{max.}=(c_1/c_2^5)c_4T^5 \tag{8}$$

and

$$Q_\lambda\text{max.}=(c_1'/c_2^4)c_4'T^4, \tag{9}$$

where $c_4$, $c_4'$, etc., represent the maximum values of $x^n/(e^x-1)$; the ratios

$$H_\lambda/H_\lambda\text{max.}=1/c_4\cdot x^5/(e^x-1), \tag{10}$$

$$Q_\lambda/Q_\lambda\text{max.}=1/c_4'\cdot x^4/(e^x-1), \tag{11}$$

$$H_\nu/H_\nu\text{max.}=1/c_4''\cdot x^3/(e^x-1)$$

and

$$Q_\nu/Q_\nu\text{max.}=1/c_4'''\cdot x^2/(e^x-1).$$

In the slide rule only the first two ratios are used since

$$H_\nu=(H_\lambda/H_\lambda\text{max.})\times H_\lambda\text{max.}\times\lambda^2,$$

and a similar expression holds for $Q_\nu$. The graphs of the four fractions are given in Fig. 1 while the values associated with the maxima are listed in Table I.

### 2. Integration formulas

Formula (3) may be integrated by parts to give

$$H_{\lambda_1-\lambda_2}=\frac{c_1}{c_2^4}T^4\left|\sum_{n=1}^{\infty}\frac{1}{n^4}[(nx)^3+3(nx)^2+6nx+6]e^{-nx}\right|_{x_2}^{x_1}$$

from which we obtain

$$H_{0-\lambda}=\frac{c_1}{c_2^4}T^4\sum_{n=1}^{\infty}\frac{1}{n^4}[(nx)^3+3(nx)^2+6nx+6]e^{-nx},$$

$$H=H_{0-\infty}=(c_1/c_2^4)T^4\times6\times(1^{-4}+2^{-4}+3^{-4}+\cdots),$$

and since

$$\sum_{r=1}^{\infty}r^{-4}=\pi^4/90$$

$$H=(c_1/c_2^4)T^4(\pi^4/15)=\sigma T^4$$
$$=5.67283\times10^{-12}T^4 \text{ watt } cm^{-2}. \tag{12}$$

The latter is the expression for the total power radiated per unit area of a black body at temperature $T$.

Further we have

$$\frac{H_{0-\lambda}}{H}=\frac{15}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}[(nx)^3+3(nx)^2+6nx+6]e^{-nx}. \tag{13}$$

For larger values of $\lambda$, $H_{0-\lambda}\simeq H$, and in such cases it is better to use

$$H_{\lambda-\infty}=H-H_{0-\lambda} \quad\text{and}\quad H_{\lambda-\infty}/H=1-H_{0-\lambda}/H. \tag{13a}$$

Similarly in terms of photons

$$Q_{0-\lambda}=\frac{c_1'}{c_2^3}T^3\sum_{n=1}^{\infty}\frac{1}{n^3}[(nx)^2+2nx+2]e^{-nx},$$

$$Q=Q_{0-\infty}=\frac{c_1'}{c_2^3}T^3\times2\times\frac{\pi^3}{25.79436\cdots}$$

$$=\sigma'T^3=1.5213\times10^{11}T^3 \text{ photon sec.}^{-1}\ cm^{-2}, \tag{14}$$

$$\frac{Q_{0-\lambda}}{Q}=\frac{12.89718\cdots}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}[(nx)^2+2nx+2]e^{-nx}, \tag{15}$$

and

$$Q_{0-\lambda}/Q=1-Q_{\lambda-\infty}/Q. \tag{15a}$$

The generalized curves for the four fractions over a range of $\lambda T$ are included in Fig. 2.

## IV. THE SCALES OF THE SLIDE RULE

The general arrangement of the scales of the slide rule may be seen in Fig. 3. For convenience the scales are lettered from *a* to *s* and these letters will be used in referring to the various scales.

### 1. Stock

Taking as a basis the value of $T$, the equations listed below are obtained by taking logarithms of both sides of (7), (8), (12), (9), and (14):

$$\log\lambda_m = \log(c_2/\beta) - \log T,$$
$$\log H_\lambda\text{max.} = \log(c_1 c_4/c_2^5) + 5\log T,$$
$$\log H = \log\sigma + 4\log T,$$
$$\log Q_\lambda\text{max.} = \log(c_1' c_4'/c_2^4) + 4\log T,$$
$$\log Q = \log\sigma' + 3\log T.$$

If $T$ is plotted as a logarithmic scale ($d$) with modulus $m_T$, the remaining scales are also logarithmic with moduli given by the second terms on the right-hand side of the equation:

$$m_\lambda = -m_T \text{ (inverse scale } e\text{)},$$
$$m_{H_\lambda\text{max}} = \tfrac{1}{5}m_T \text{ (scale } b\text{)}, \quad m_H = \tfrac{1}{4}m_T \text{ (scale } a\text{)},$$
$$m_{Q_\lambda\text{max}} = \tfrac{1}{4}m_T \text{ (scale } g\text{)}, \quad m_Q = \tfrac{1}{3}m_T \text{ (scale } f\text{)}.$$

The constants in the various equations define the positions of the corresponding scales of the stock relative to the temperature scale. These constants are of two types, firstly those of purely mathematical conception and so invariant (e.g., $\beta$, $c_4$, $c_4'$), and secondly those derived from physical constants ($c_1$, $c_2$, $c_1'$) which will depend on the values taken for $c$, $h$, and $k$. The latter define the relative positions of the scales on the stock, the former those on the slide. The values of the constants used have already been listed. The procedure necessary for the use of the slide rule with changed values of the constants will be given later.

Since in practical problems temperatures are usually expressed in degrees centigrade, the appropriate scale has been included ($c$). It has been assumed that 0°C = 273.18°K.

In the construction of the slide rule, a modulus for the temperature scale has been chosen $m_T = 20$ cm, and about two decades taken covering directly the range 100°K to 10,000°K (−180°C to about 10,000°C). The same scale of wave-length is used to cover the two ranges, 0.3 to 30 microns (black figures) and 30 to 3000 microns (3 mm) (red figures).

All the scales on the stock, with the exception of scale $e$, are purely logarithmic, differing only by the moduli.

### 2. Slide

The slide is double-sided, one side (energy) for use in the $H_{\lambda_1-\lambda_2}$, $H_\lambda$, and $H_\nu$ calculations, the other (photon) for $Q_{\lambda_1-\lambda_2}$, $Q_\lambda$, and $Q_\nu$. Since the two sides are similar, differing only quantitatively, only one will be discussed in detail.

As the slide scales are obtained from the fractions (13), (13a), (15), and (15a) in which the physical constants are eliminated, the scales themselves are independent of the values chosen for these constants. The scales are also independent of the stock except in so far as the moduli are determined by $m_T$. The two upper scales ($h$ and $i$) are those of the fraction $H_\lambda/H_\lambda$max. for the two-wave-length ranges. For $x=\beta$, $H_\lambda/H_\lambda$max. = 1 and at this point lines marked "$H_\lambda$max." and "Temperature" have been added to the slide. If the slide is moved so that the latter line is opposite a given temperature, then the former line is opposite the corresponding value of $\lambda_m$ on scale $e$. Scale $h$ or $i$ gives the values of $H_\lambda/H_\lambda$max. corresponding to any other value of $\lambda$. Three lines have been added to scale $h$ giving also the wave-lengths corresponding to the maximum values of $H_\nu$, $Q_\lambda$, and $Q_\nu$.

If scale $e$ is extended to longer wave-lengths by two decades, the corresponding values of scale $h$ go down to $10^{-11}$. The extended wave-length scale may be displaced to coincide with the original scale with values of $\lambda$ 100 times larger, as given by the red figures. The appropriately displaced scale of $H_\lambda/H_\lambda$max., $i$, is also colored red. The fact that, in the direction of longer wave-lengths, scale $i$ approximates to a logarithmic scale of modulus $\tfrac{1}{4}m_T = \tfrac{1}{4}m_\lambda$, permits the extension of the rule in this direction.

In the same way scale $m$ (black) covers the fraction $H_{0-\lambda}/H$ (13) for the shorter wave-length range, and scale $l$ (red) the fraction $H_{\lambda-\infty}/H$ (13a) for the longer range. Scale $l$ approximates to a logarithmic scale of modulus $\tfrac{1}{3}m_T$.

On the photon side of the slide are the corresponding scales, $n$ and $o$, of $Q_\lambda/Q_\lambda$max. (11) together with the scales, $s$ and $r$, of $Q_{0-\lambda}/Q$ (15) and $Q_{\lambda-\infty}/Q$ (15a) respectively. Scales $o$ and $r$ approximate to logarithmic scales of moduli $\tfrac{1}{3}m_T$ and $\tfrac{1}{2}m_T$, respectively. The five lines mentioned above are, of course, repeated on the photon side of the slide.

In deriving the positions of the scale lines, involved fractions were calculated for round values of argument $x$, and then, with the aid of Interpolation Tables (HMSO, 1936), were inversely interpolated for round values of fractions using Bessel's interpolation formula including fourth and sometimes sixth differences. All calculations were made with two Brunsviga calculating machines by the method of balancing approximation.

For convenience in certain calculations, two logarithmic scales are included on each side of the slide, $j$ and $k$ on the energy side with moduli $\tfrac{1}{5}m_T$ and $\tfrac{1}{4}m_T$, and $p$ and $q$ on the photon side with moduli $\tfrac{1}{4}m_T$ and $\tfrac{1}{3}m_T$.

### 3. Back of stock

Scales relating wave-length, wave number, and electron volts are included on the back of the stock. For convenience in reading from these scales without

TABLE II. Accuracy of calculations using slide rule.

| | Long wave-lengths | Vicinity of maximum spectral value | Short wave-lengths Values of fractions $\sim 10^{-3}$ | $10^{-6}$–$10^{-25}$ |
|---|---|---|---|---|
| $H_\lambda/H_\lambda$max. | 1.5% | ~0.5% | | |
| $H_{\lambda-\infty}/H$ or $H_{0-\lambda}/H$ | 1.25% | 1.25% | | |
| | | | 3.0% | 5.0% |
| $Q_\lambda/Q_\lambda$max. | 1.25% | ~0.5% | | |
| $Q_{\lambda-\infty}/Q$ or $Q_{0-\lambda}/Q$ | 1.0% | 1.0% | | |

the use of the cursor, the wave number scale is given twice.

## V. USE OF THE SLIDE RULE

In following the description of the use of the slide rule reference may be made to Fig. 3. The values given in brackets below refer to the actual settings of the scales in the figure. A straight edge should be used in place of a cursor.

(a) If the cursor line is set on a given value of temperature on scale $c$ or $d$ (say 727°C or 1000°K), then the following data may be read directly from the cursor line:

Scale $e$—wave-length $\lambda_m$ (2.9 microns) corresponding to the maximum spectral radiant flux density, $H_\lambda$max.;
Scale $a$—total radiant flux density, $H$ (5.7 watt $cm^{-2}$);
Scale $b$—maximum spectral radiant flux density, $H_\lambda$max. ($1.3\times10^4$ watt $cm^{-2}$ $\mathrm{cm}_{\Delta\lambda}^{-1}$);
Scale $f$—total photon flux density, $Q$ (1.52 photon sec.$^{-1}$ $cm^{-2}$);
Scale $g$—maximum spectral photon flux density, $Q_\lambda$max. ($2.08\times10^{23}$ photon sec.$^{-1}$ $cm^{-2}$ $\mathrm{cm}_{\Delta\lambda}^{-1}$).

(b) If the line marked "Temperature" is brought into coincidence with a given temperature (1000°K), and the cursor line placed against a given wave-length (say 1 micron or 100 microns), the values of the following fractions may be read directly from the cursor line:

For $\lambda=0.3-30$ microns:
Scale $h$—$H_\lambda/H_\lambda$max.($1.6\times10^{-2}$); Scale $m$—$H_{0-\lambda}/H$($3.3\times10^{-4}$);
Scale $n$—$Q_\lambda/Q_\lambda$max.; Scale $s$—$Q_{0-\lambda}/Q$;
For $\lambda=30-3000$ microns:
Scale $i$—$H_\lambda/H_\lambda$max.($1.9\times10^{-5}$); Scale $l$—$H_{\lambda-\infty}/H$($1.45\times10^{-4}$);
Scale $o$—$Q_\lambda/Q_\lambda$max.; Scale $r$—$Q_{\lambda-\infty}/Q$.

Further quantities may then be estimated using, if desired, the appropriate multiplication scales:

$$H_\lambda=(H_\lambda/H_\lambda\mathrm{max.})\times H_\lambda\mathrm{max.}(2.08\times10^2\ \mathrm{watt}\ cm^{-2}\ \mathrm{cm}_{\Delta\lambda}^{-1}),$$
$$H_\nu=(H_\lambda/H_\lambda\mathrm{max.})\times H_\lambda\mathrm{max.}\times\lambda_\mu^2\times10^{-8}(2.08\times10^{-6}\ \mathrm{watt}\ cm^{-2}\ (\mathrm{cm}_{\Delta\nu}^{-1})^{-1}),$$
$$H_{\lambda_1-\lambda_2}=[(H_{0-\lambda_1}/H)-(H_{0-\lambda_2}/H)]\times H=[(H_{\lambda_1-\infty}/H)-(H_{\lambda_2-\infty}/H)]\times H,$$
$$Q_\lambda=(Q_\lambda/Q_\lambda\mathrm{max.})\times Q_\lambda\mathrm{max.}$$
$$Q_\nu=(Q_\lambda/Q_\lambda\mathrm{max.})\times Q_\lambda\mathrm{max.}\times\lambda_\mu^2\times10^{-8},$$
$$Q_{\lambda_1-\lambda_2}=[(Q_{0-\lambda_1}/Q)-(Q_{0-\lambda_2}/Q)]\times Q=[(Q_{\lambda_1-\infty}/Q)-(Q_{\lambda_2-\infty}/Q)]\times Q.$$

In the formulas with square brackets only positive values of differences are taken. Both values of the fractions inside the bracket must be read from the same scale, or, if this is not possible since one wave-length is within the black range and the other within the red, then one of the fractions must be subtracted from unity.

The multiplication scales are so arranged that it is unnecessary to move the slide, when set at a given temperature, in order to carry out the operation of multiplication. Care must be taken, however, to use the correct multiplication scale for a given operation, as marked on the slide. When calculations of $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ are being carried out for a narrow wave-length interval, the accuracy falls with decreasing interval, but may be increased by calculating the spectral values of $H_\lambda$, $H_\nu$, $Q_\lambda$, or $Q_\nu$ for the middle point of the interval, and, assuming the spectral quantity to be constant over the interval, multiplying the quantity by the width of the interval.

## VI. EXTENSION RULES AND CHANGE OF CONSTANTS

### 1. Temperature

The slide rule may be used for calculations involving temperatures outside the range 100°K to 10,000°K by a simple extension process. The temperature $T$ is expressed in degrees absolute as $T_s=T\times10^n$, where $n$ is a convenient integer such that $T_s$ is within the compass of scale $d$. The corresponding wave-length $\lambda_s=\lambda\times10^{-n}$. By the processes already described, the values of the various quantities $B_s$ are found for temperature $T_s$ and wave-length $\lambda_s$, and the values appropriate to the original temperature obtained from the following formulas (the corresponding factors are given on the back of the rule):

$$H_\lambda=(H_\lambda)_s\times10^{-5n},\qquad Q_\lambda=(Q_\lambda)_s\times10^{-4n},$$
$$H_\nu=(H_\nu)_s\times10^{-3n},\qquad Q_\nu=(Q_\nu)_s\times10^{-2n},$$
$$H_{\lambda_1-\lambda_2}=(H_{\lambda_1-\lambda_2})_s\times10^{-4n},\qquad Q_{\lambda_1-\lambda_2}=(Q_{\lambda_1-\lambda_2})_s\times10^{-3n}.$$

A similar process may be applied to problems in which, although $T$ is inside the normal range, the calculations move outside the limits of the slide.

### 2. Long wave-lengths

Since the red scales covering the longer wave-length range to 3000 microns ($i$, $l$, $o$, $r$) approximate to iogarithmic, the slide rule may be extended to longer wave-lengths. The wave-length is replaced by a wave-length $\lambda_s$, such that $\lambda_s=\lambda\times10^{-n}$, where $n$ is the smallest possible positive integer that will bring $\lambda_s$ inside the normal red wave-length range of the rule. The corresponding quantities $B_s$ may be calculated and transposed to the values for the original wave-length using the formulas given for the extension of the temperature scales.

### 3. Change of constants

From formulas (1) to (6) it can be seen that:

(a) the constant $c_2$ appears only in the ratio $T/c_2$, and
(b) the quantities related to energy and photons are proportional to $c_1$ and $c_1'$, respectively.

Thus the use of the slide rule with different values of the physical constants is simple. If it is desired to introduce new values ${}_nc_1$, ${}_nc_2$, and ${}_nc_1'$, it is only necessary to make all calculations for a temperature $T_n$ where $T_n = T \times c_2/{}_nc_2$ and multiply the values so obtained by the following factors:

$$\text{Energy} \quad {}_nc_1/c_1 = h'c'^2/hc^2$$
$$\text{Photons} \quad {}_nc_1'/c_1' = c'/c$$

where $c'$ and $h'$ are the new values of the constants.

## VII. ACCURACY

The accuracy with which calculations can be made is limited by the accuracy with which the scales can be read. For any logarithmic scale the error is constant and approximately inversely proportional to the modulus of the scale. In the case of the scales of $H_\lambda$max.($b$), $H(a)$, $Q_\lambda$max.($g$), and $Q(f)$ (the scales with the smaller moduli) the error can be eliminated if desired by using the exact formulas given in Section II and included on the back of the rule:

$$H_\lambda\text{max.} = 1.2875 \times 10^{-11} T^5, \quad H = 5.6728 \times 10^{-12} T^4,$$
$$Q_\lambda\text{max.} = 2.1027 \times 10^{11} T^4, \quad Q = 1.5213 \times 10^{11} T^3.$$

The logarithmic scales limiting the accuracy of any calculations are those of temperature and wave-length. Both scales are of modulus 20 cm and the accuracy of reading is of the order of 0.25 percent. The other scales involved are those of the fractions ($h$, $i$, $l$, $m$, $n$, $o$, $r$, $s$) which are not logarithmic. In general, the relative accuracy will vary according to the region of the scale in use. However, for the longer wave-lengths (red scales) the errors are constant, the values being:

$H_\lambda/H_\lambda$max. ($i$)—about 1 percent
$H_{\lambda-\infty}/H$ and $Q_\lambda/Q_\lambda$max. ($l$ and $o$)—about 0.75 percent
$Q_{\lambda-\infty}/Q$ ($r$)—about 0.5 percent

In the region of the respective maxima, the relative error for the fractions is reduced to a few tenths of one

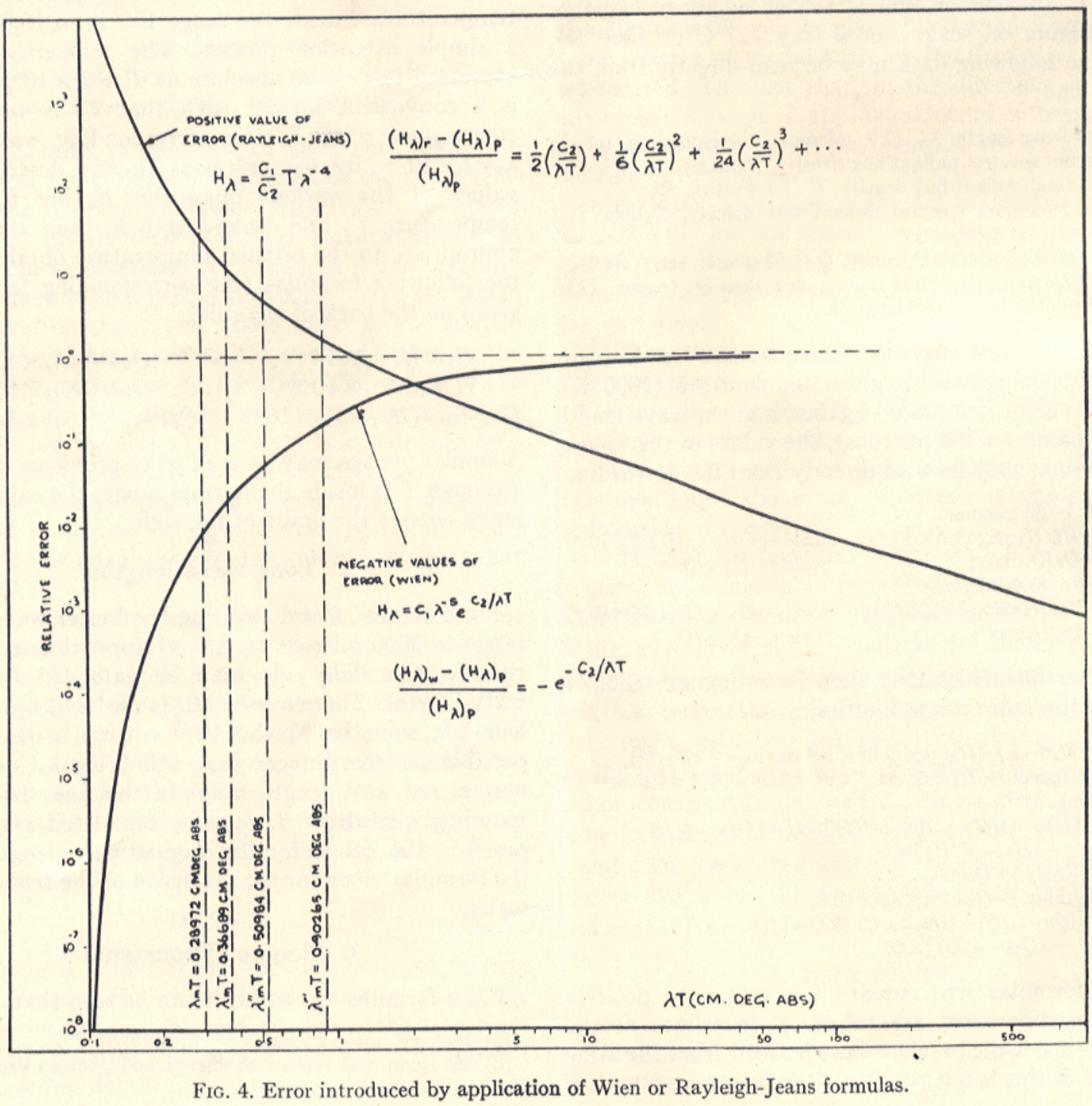


FIG. 4. Error introduced by application of Wien or Rayleigh-Jeans formulas.

percent, and increases from this value as shown in Table II, being approximately the same for all scales. The introduction of the diagonal scale presentation for the short wave-length ratio scales at one end of the slide prevents serious reduction of accuracy at the shortest wave-lengths.

In $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ calculations the relative error is increased by taking differences, the error being greater for a narrow interval. For this reason it is better to assume a constant value of $H_\lambda$ or $Q_\lambda$ over the interval as described in Section V.

## VIII. COMPARISON OF THE PLANCKIAN FORMULA WITH WIEN AND RAYLEIGH-JEANS FORMULAS

Both the Wien and Rayleigh-Jeans formulas may be treated as mathematical approximations of the exact Planck formula. The latter may be written in the form

$$(H_\lambda)_p T^{-5} = (c_1/c_2^5)x^5/(e^x-1).$$

It is possible to deduce either the Wien formula or the Rayleigh-Jeans formula and obtain the errors introduced by the use of these formulas in terms of $x$ or $T$.

If $x \gg 1$ ($\lambda T$ very small) then $e^x$ is sufficiently larger than unity for $e^x-1$ to be replaced by $e^x$, when

$$(H_\lambda)_w T^{-5} = (c_1/c_2^5)(x^5/e^x).$$

This is Wien's formula which is usually written in the form

$$(H_\lambda)_w = c_1\lambda^{-5}e^{-c_2/\lambda T}.$$

The error involved in using this approximation is

$$E_w = \frac{(H_\lambda)_w-(H_\lambda)_p}{(H_\lambda)_p} = \frac{x^5e^{-x}-x^5/(e^x-1)}{x^5/(e^x-1)}$$

$$= e^{-x}(e^x-1)-1 = e^{-x} = e^{-c_2/\lambda T}.$$

If, on the other hand, $x \ll 1$ ($\lambda T$ very large) then

$$e^x-1 = (1+x+\tfrac{1}{2}x^2+\cdots)-1 = x+\tfrac{1}{2}x^2+\cdots \simeq x$$

and

$$(H_\lambda)_r T^{-5} = (c_1/c_2^5)x^4.$$

This is, of course, the Rayleigh-Jeans formula and is usually written in the form

$$(H_\lambda)_r = (c_1/c_2)T\lambda^{-4}.$$

The relative error is

$$E_r = \frac{(H_\lambda)_r-(H_\lambda)_p}{(H_\lambda)_p} = \frac{x^4-x^5/(e^x-1)}{x^5/(e^x-1)} = \frac{e^x-1-x}{x}$$

$$= \frac{x}{2}+\frac{x^2}{6}+\cdots = \frac{1}{2}c_2/\lambda T+\frac{1}{6}(c_2/\lambda T)^2+\cdots$$

The corresponding formulas for $H_\nu$, $Q_\lambda$, and $Q_\nu$ introduce similar errors as a function of $x$.

In the case of calculations involving $H_{\lambda_1-\lambda_2}$ or $Q_{\lambda_1-\lambda_2}$ estimation of the error is more complicated since the error is equivalent to some intermediate value of the error in the spectral quantity within the wave-length range involved. It can, however, be shown that the above formulas may be used to give the upper limit of the error if the largest value of $\lambda T$ (smallest value of $x$) in the interval is substituted in the expression for the error in the case of the Wien formula, or the smallest value of $\lambda T$ in the case of the Rayleigh-Jeans formula.

Figure 4 represents errors for both of the simplified formulas as a function of $\lambda T$. The errors introduced by using either formula at given values of $\lambda$ or $\lambda T$ can be estimated using this graph.

## ACKNOWLEDGMENTS

The author wishes to acknowledge the valuable suggestions made in the course of many discussions by Dr. E. Lee, who initiated the work, Dr. A. Elliott, and Mr. T. H. Pratt, and is particularly grateful to Mr. L. A. J. Verra for assistance throughout the lengthy process of computing. In addition, the assistance of the Mathematics Division, N.P.L., in checking the calculations for the fractional scales was much appreciated.

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*Reprinted from the article in The Review of Scientific Instruments by M. W. Makowski.*